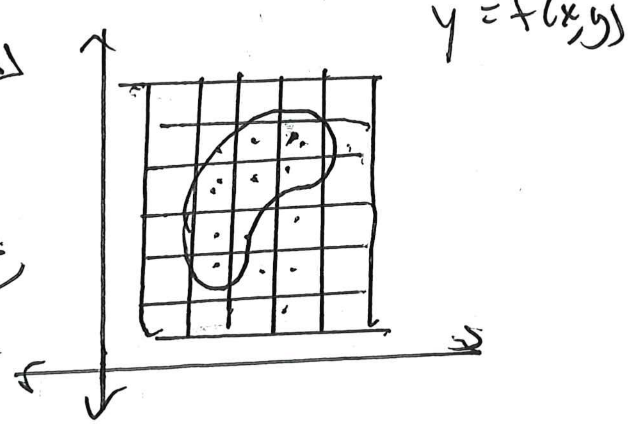
PROJECTED WRITTEN NOTES FROM THE M 408D LECTURE ON TUESDAY, APRIL 23, 2024, ON SECTION 15.5 — ON CALCULATING SURFACE AREA ON THE SURFACE GRAPH OF A FUNCTION

CLASS # 27

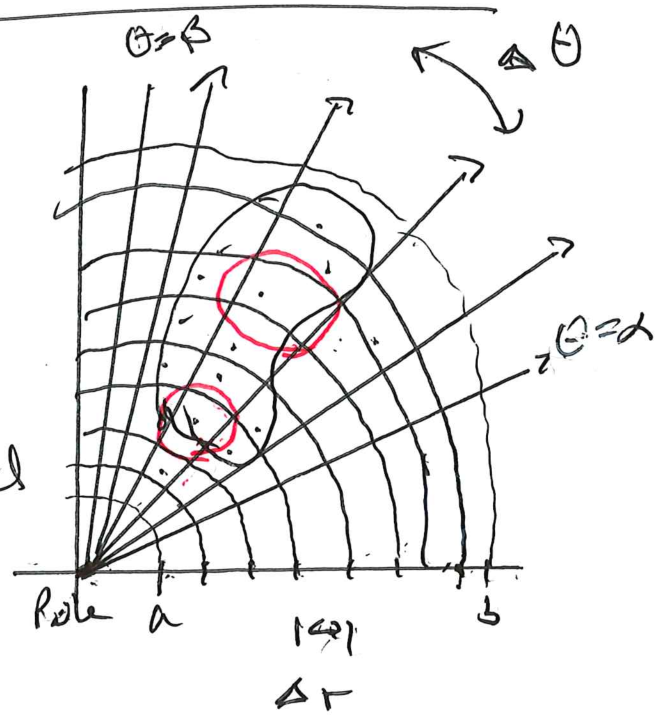
For Double Integrals using Polar Coordinates, why is there a special r factor?

For $\iint_D f(x,y)\,dA$

In Rectangular Double Riemann Sums, All Subrectangles have the same area

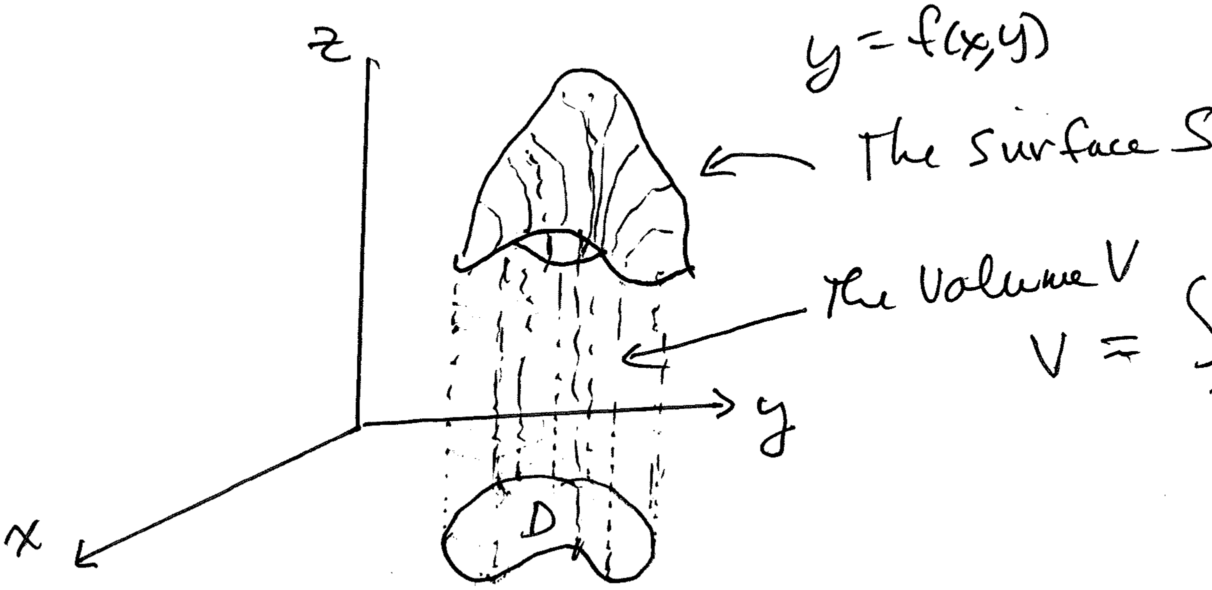


In Polar Riemann Sums, The Areas of the Subrectangs Grow proportional to the distance from The Pole, by a factor of r.



Sec. 15.5: The surface area $A(S)$ of a surface S on a surface graph.

Given function $z = f(x,y)$ and a region D in the x,y plane, The surface S is that part of the surface graph of f that lies above region D.



$$V = \iint_D f(x,y)\,dA$$

The Surface Area $A(S)$ is given by

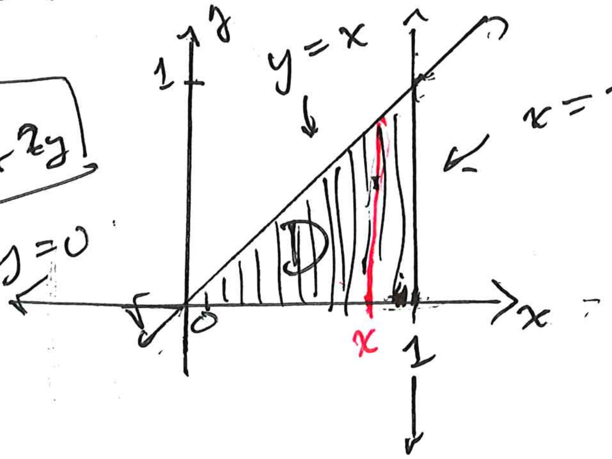
$$A(S) = \iint_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\; dA$$

$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA.$$

PROBLEM: Let $z = f(x, y) = x^2 + 2y$ and

Let Region D be the region bounded by $y = x$, $x = 1$ and the x-axis, $y = 0$.

$\boxed{f(x,y) = x^2 + 2y}$



Type I description:

$D: \; 0 \le x \le 1$
$\quad\; 0 \le y \le x$

S is the part of the graph of f that is above D.

Find $A(S)$.

Sol'n: $z = f(x,y) = x^2 + 2y$, $f_x = 2x$, $f_y = 2$

$$A(S) = \iint_D \sqrt{(2x)^2 + (2)^2 + 1} \, dA$$

$$A(S) = \iint_D \sqrt{4x^2 + 5} \, dA$$

$$= \int_0^1 \left(\int_0^x \sqrt{4x^2 + 5} \, dy \right) dx.$$

$$= \int_0^1 \left(\left(y \cdot \sqrt{4x^2 + 5} \right) \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_0^1 \left(\left(x\sqrt{4x^2+5} \right) - (0) \right) dx$$

$$= \int_0^1 \left(x\sqrt{4x^2+5} \right) dx$$

let $u = 4x^2+5$

$du = 8x\,dx$

$x\,dx = \frac{1}{8} du$

when $x = 0$, $u = 5$

when $x = 1$, $u = 9$

$$= \frac{1}{8} \int_5^9 u^{\frac{1}{2}} du$$

$$= \frac{1}{8} \left(\frac{2}{3} u\sqrt{u} \right) \Big|_5^9 \quad u^{3/2}$$

$$= \frac{1}{8} \left(\frac{2}{3} 9\sqrt{9} - \frac{2}{3} 5\sqrt{5} \right)$$

$$= \frac{1}{12}(27) - \frac{1}{12} 5\sqrt{5}$$

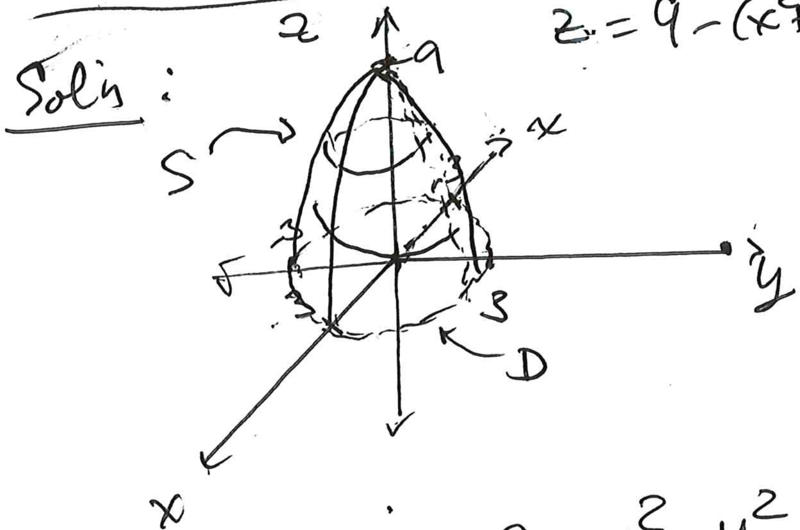
$$A(S) = \frac{1}{2}(27 - 5\sqrt{5}) \text{ sq units}$$

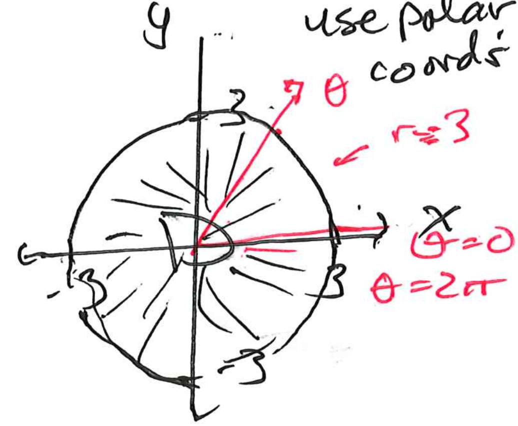
Problem: $z = f(x,y) = 9-(x^2+y^2)$

Surface S is that part of the graph of f that is above the x,y plane, $(z=0)$.

Find $A(S)$.

Sol'n:





$$z = 9 - x^2 - y^2$$

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y$$

$$A(S) = \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\; dA$$

$$= \iint_D \sqrt{4x^2 + 4y^2 + 1}\; dA$$

$$= \iint_D \sqrt{4(x^2+y^2) + 1}\; dA$$

$$= \int_0^{2\pi}\left(\int_0^3 \left(\sqrt{4r^2+1}\right)\cdot r\, dr\right) d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{8}\int_1^{37} u^{\frac{1}{2}}\, du\right) d\theta$$

let $u = 4r^2+1$

$du = 8\, r\, dr$

$r\, dr = \frac{1}{8} du$

when $r=0$, $u=1$

when $r=3$, $u=37$

$u^{3/2}$

$$= \frac{1}{8}\int_0^{2\pi}\left(\left(\frac{2}{3} u\sqrt{u}\right)\Big|_{u=1}^{37}\right) d\theta$$

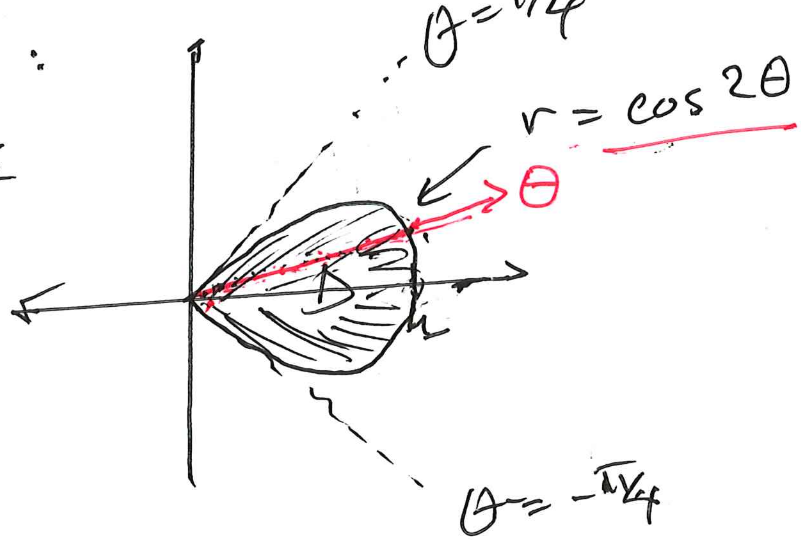
$= \text{WORK} \ldots \text{WORK} \ldots = \frac{\pi}{6}\left(37\sqrt{37}-1\right).$

$A(S) = \frac{\pi}{6}\left(37\sqrt{37}-1\right)$ sq units

Problem: Over the Region D inside the petal $r = \cos 2\theta$, $-\pi/4 \le \theta \le \pi/4$, determine $\iint_D \sqrt{x^2+y^2}\, dA$.

Soln:

$f(x,y) = \sqrt{x^2+y^2}$



$$\iint_D \sqrt{x^2+y^2}\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} \sqrt{r^2} \cdot r \cdot dr\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^2\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\left(\frac{1}{3} r^3\right) \Bigg|_{r=0}^{r=\cos 2\theta} \right) d\theta$$

$$\int_{-\pi/4}^{\pi/4} \left(\tfrac{1}{3} \cos^3 2\theta \right) d\theta = \dots = \frac{2}{9}$$

$$\iint_D \sqrt{x^2+y^2}\, dA = 2/9$$

petal